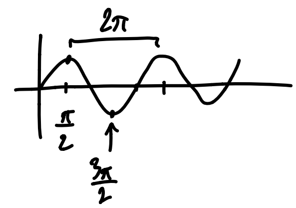
WORKSHEET
DISCUSSION SECTION 7
DUE 5/20 AT MIDNIGHT PDT

(1) Consider the function

$$f(x,y) = \sin\left(\frac{1}{x^2+y^2}\right).$$



(a) Describe the behavior of r^2, $1/r^2$, and $\sin(1/r^2)$ as r approaches 0.
(b) Describe the graph of $f(x,y)$ near $(0,0)$.
(c) Does $f(x,y)$ have a limit at $(0,0)$? Why or why not?
(d) Now consider the function

$$g(x,y) = (x^2+y^2)\cdot f(x,y).$$

Repeat parts (b) and (c) for $g(x,y)$.
(e) Now consider the function

$$h(x,y) = \frac{x^2}{x^2+y^2}\cdot f(x,y).$$

Repeat parts (b) and (c) for $h(x,y)$.

(2) Suppose $f(x,y)$ is a function and let $F(x) = f(x,b)$ for some fixed constant b. Use the limit definitions of derivatives and partial derivatives to express the partial derivative $f_x(a,b)$ in terms of $F(x)$.

(3) Let $f(x,y) = x^2 - y^2$ and let $\mathcal{G}$ be its graph.
(a) What kind of surface is $\mathcal{G}$?
(b) Parametrize the intersection of $\mathcal{G}$ with the plane $y = 2$.
(c) Take the derivative of the parametrization you found in (b) to find a tangent vector to $\mathcal{G}$ at $(1,2,f(1,2))$.
(d) Repeat the previous two parts for the intersection of $\mathcal{G}$ with the plane $x = 1$.
(e) Find a normal vector $\vec{n}$ to $\mathcal{G}$ at $(1,2,f(1,2))$.
(f) What is an equation for the tangent plane to $\mathcal{G}$ at $(1,2,f(1,2))$?

1a. r^2 : parabolic
r^{-2} : vert. asymptote
$\sin r^{-2}$: oscillation with frequency $\to \infty$ near 0.
b. Same oscillation, but the function is radially symmetric.
c. No, for the same reason as in 1D.
d. Oscillates but decays to 0 as $r \to 0$

$$\left|(x^2+y^2)\sin\frac{1}{x^2+y^2}\right| = \left|r^2 \sin\frac{1}{r}\right| \le r^2 \to 0$$

so by the squeeze lemma, the limit is 0.
e. Along $y = 0$, similar to (b). Along $x = 0$, constant $\equiv 0$.
because of this, the limit doesn't exist.

Explicitly, we can set $r_n = \sqrt{\frac{1}{\pi/2 + 2\pi n}}$
so $\frac{1}{r_n^2} = \frac{\pi}{2} + 2\pi n$,
hence $r_n \searrow 0$ but $\sin\frac{1}{r_n^2} = 1$, and similarly $s_n = \sqrt{\frac{1}{\frac{3\pi}{2} + 2\pi n}}$
satisfies $s_n \searrow 0$ but $\sin\frac{1}{s_n^2} = -1$. Since these sequential limits differ, $\lim_{r\to 0} \sin\frac{1}{r^2}$ does not exist. (see also: the fig. above)

2. $F(x) = f(x, b)$. Then $f_x(a, b) = F'(a)$:

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} = \lim_{h \to 0} \frac{F(a+h) - F(a)}{h} = F'(a).$$

3a. hyperbolic paraboloid

b. $f(x, 2) = x^2 - 4$ so we can parameterize this by

$\vec{r}(t) = (t, 2, t^2 - 4) \quad (t \in \mathbb{R})$

c. $\vec{r}'(t) = (1, 0, 2t)$

$\vec{r}'(1) = (1, 0, 2)$

d. $\vec{q}(t) = (1, t, 1 - t^2)$

$\vec{q}'(t) = (0, 1, -2t)$

$\vec{q}'(2) = (0, 1, -4)$

e. $\vec{n} = \vec{r}'(2) \times \vec{q}'(1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 0 & 1 & -4 \end{vmatrix} = (-2, +4, 1)$

f. $-2x + 4y + z = d$

$-2 \cdot 1 + 4 \cdot 2 + (-3) = d \Rightarrow d = 3.$

Alternatively,

$F = x^2 - y^2 - z$

$\nabla F = (2x, -2y, -1)$

$\vec{n} = \nabla F(1, 2) = (2, -4, -1)$ (we'll see this method later on)