Spectrum Analysis of a Block Random Matrix Model

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Wigner matrices

Definition

We say H is a *Wigner matrix* if it is an $N \times N$ Hermitian matrix whose entries are complex random variables satisfying:

1. The upper-triangular entries $(H_{ij})_{1 \leq i \leq j \leq N}$ are iid
2. For all i, j: $\mathbb{E}H_{ij} = 0$ and $\mathbb{E}|H_{ij}|^2 = N^{-1}$
3. $\mathbb{E}|\sqrt{N}H_{ij}|^p \leq C_p$ for any $p \geq 1$, uniformly in N, i, j

- The (normalized) eigenvectors $\mathbf{v}_k$ of a Wigner matrix are *completely delocalized* in the sense that

$$\max_i |(\mathbf{v}_k)_i| \leq N^{-1/2+\epsilon}$$

with high probability

- That is, the mass of each eigenvector is spread approximately evenly over all entries

Localization and delocalization

- The eigenvectors of Wigner matrices are delocalized
- The eigenvectors of the model

$$\begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}, \quad (H_1, H_2 \text{ indep. Wigner matrices})$$

  are "localized": all of the mass of each eigenvector is concentrated in its first half or second half.

- One-dimensional random band matrices with band with W and size N are conjectured to exhibit a phase transition at the scale $W_c = \sqrt{N}$: i.e. eigenvectors are localized for $W \ll W_c$ and delocalized for $W \gg W_c$.

The block model

Suppose we have D iid interacting systems. How strong must the interaction be for mixing to occur?

If $D \geq 3$, we define

$$H_\Lambda = H + \Lambda := \begin{bmatrix} H_1 & A & 0 & \cdots & A^* \\ A^* & H_2 & A & 0 & 0 \\ 0 & A^* & H_3 & A & \vdots \\ \vdots & 0 & A^* & \ddots & A \\ A & 0 & \cdots & A^* & H_D \end{bmatrix} \quad (1)$$

where H_j are iid $N \times N$ Wigner matrices, and if $D = 2$, we similarly define

$$H + \Lambda := \begin{bmatrix} H_1 & A \\ A^* & H_2 \end{bmatrix} \quad (2)$$

Notation and assumptions

If $a, b \in \mathbb{R}$, we will write $[\![a, b]\!] = [a, b] \cap \mathbb{Z}$, $\quad [\![b]\!] = [\![1, b]\!]$. Define

$$\mathcal{I} := [\![DN]\!], \quad \mathcal{I}_a = [\![(a-1)N + 1, aN]\!], \quad (a \in [\![D]\!]).$$

Define block restrictions of the identity matrix by

$$(E_a)_{ij} = \mathbf{1}_{i=j, i \in \mathcal{I}_a}, \quad (a \in [\![D]\!]).$$

We will write $\|B\|_{HS}^2 = \sum_{i,j} |B_{ij}|^2$ for the Hilbert-Schmidt norm and $\langle B \rangle = (DN)^{-1} \operatorname{Tr} B$ for the normalized trace of a $DN \times DN$ matrix.

Assumption ($\bigstar$)

Let $A \in \mathbb{C}^{N \times N}$ be a deterministic matrix s.t. $\|A\|_{op} \leq N^{-\delta_A}$ for a constant $\delta_A > 0$, and let $H_1, \ldots, H_D$ be D independent $N \times N$ iid regular Wigner matrices. Then we consider the model $H_\Lambda := H + \Lambda$, defined by (1) and (2).
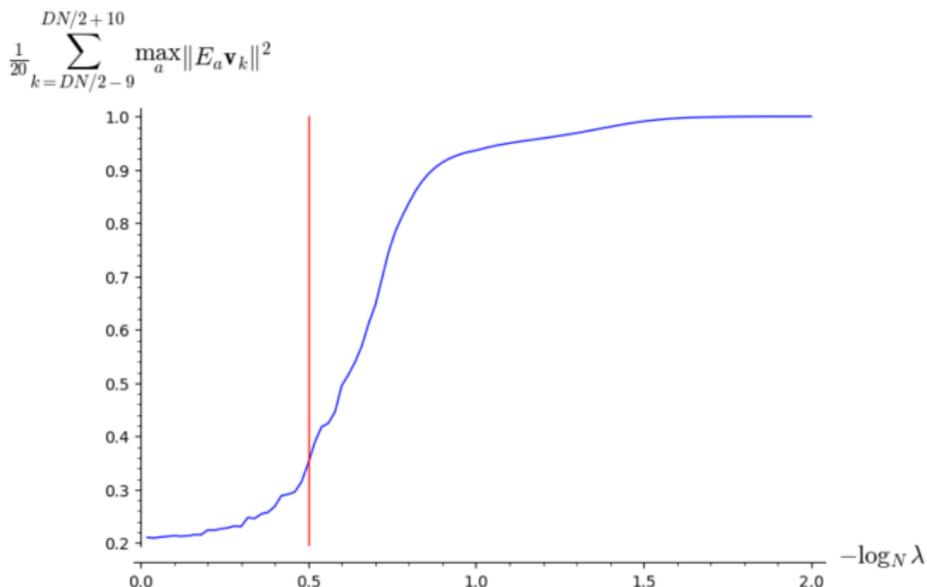
Example: single bulk eigenvector

We say $\mathbf{v}_k$ is a bulk eigenvector if $\kappa DN \leq k \leq (1 - \kappa)DN$, $\kappa > 0$. We plot $|(\mathbf{v}_{DN/2+1})_j|^2$ vs j, where $A = \lambda I$, $\lambda = N^{-0.2}, N^{-0.4}, N^{-0.6}, N^{-0.8}$:
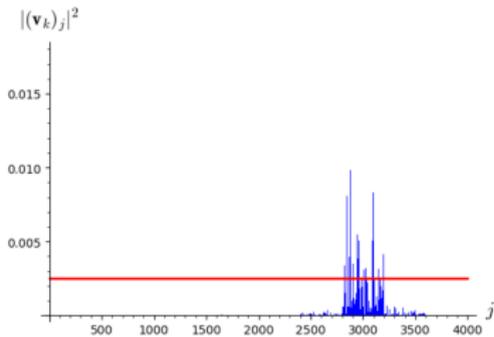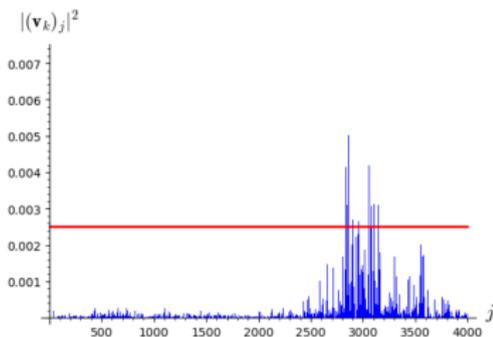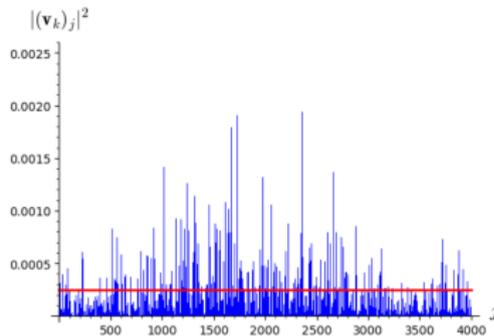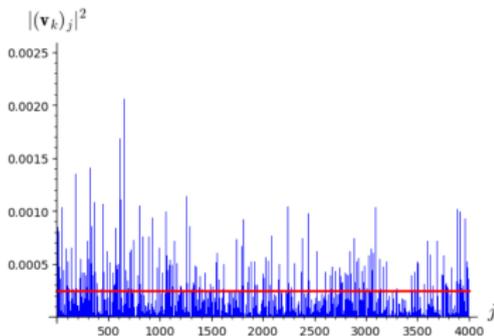
Example: bulk

Let $D = 5, N = 400, A = \lambda I$ where $\lambda \in \mathbb{R}$, and assume H is a GUE. We plot the average of $\max_a \|E_a \mathbf{v}_k\|^2$ for 20 bulk eigenvectors with $k \approx DN/2$:

$$\frac{1}{20} \sum_{k=DN/2-9}^{DN/2+10} \max_a \|E_a \mathbf{v}_k\|^2$$

Example: edge

The threshold depends on the distance to the edge. We plot $\max_a \|E_a \mathbf{v}_{DN}\|^2$ for the largest eigenvector $k = DN$:
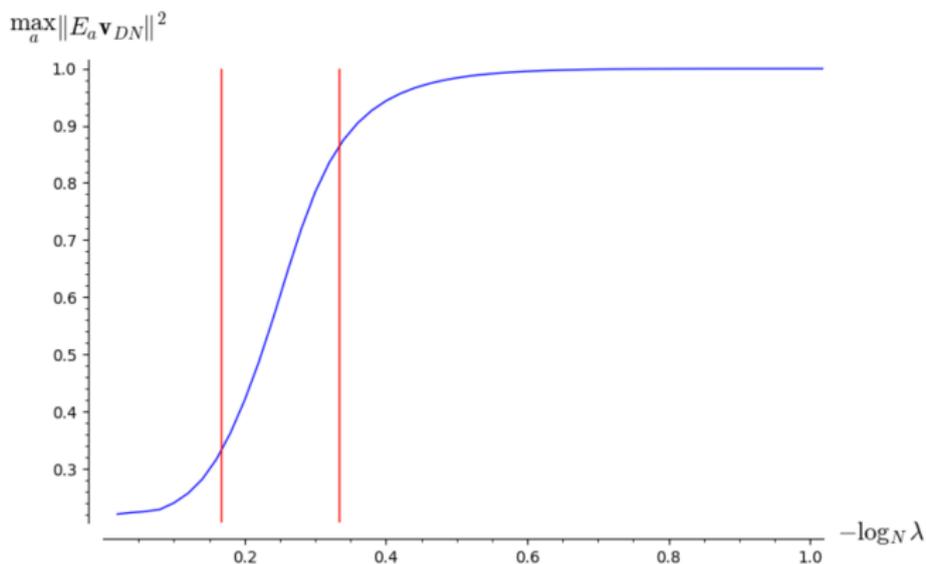
Delocalization

Theorem

Suppose that $k \in [\![DN/2]\!]$, Assumption ★ holds, and A satisfies

$$\|A\|_{HS} \geq N^{1/3+\epsilon_A} k^{-1/3} \tag{3}$$

for some $\epsilon_A > 0$. Then there exists $c > 0$ such that the eigenvector $\mathbf{v}_k$ of H_Λ satsifies

$$\max_{a \in [\![D]\!]} \left| \mathbf{v}_k^* E_a \mathbf{v}_k - D^{-1} \right| \leq N^{-c} \tag{4}$$

with high probability.

Localization

Theorem

Suppose that H is a Gaussian Unitary Ensemble, $k = O(1)$ (i.e. the bound is independent of N), and A satisfies Assumption ★ and the bound

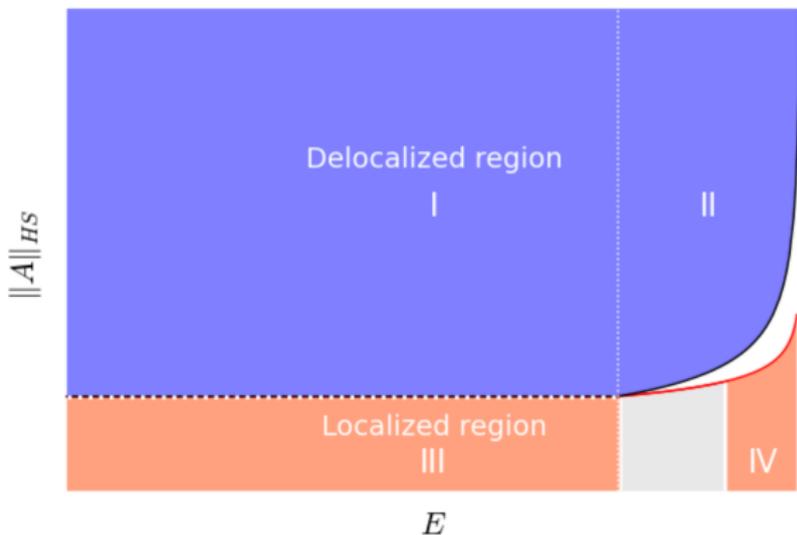$$\|A\|_{HS} \leq N^{1/6 - \epsilon_A} k^{-1/6} \tag{5}$$

for some $\epsilon_A > 0$. Then there exists $c > 0$ such that

$$\max_{a \in [\![D]\!]} |\mathbf{v}_k^* E_a \mathbf{v}_k| \geq 1 - N^{-c} \tag{6}$$

with high probability.
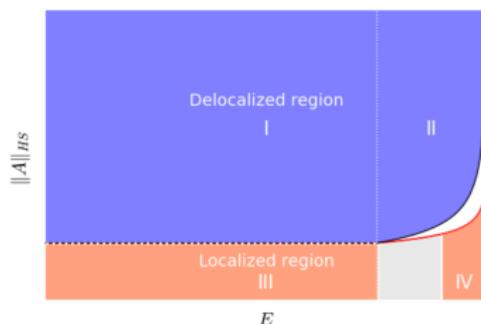
Summary of results

- Region I: delocalization in bulk for $\|A\|_{HS} \geq N^{\epsilon_A}$ [S. Yang Yin '24]
- Region II: delocalization near edge for $\|A\|_{HS} \geq N^{1/3+\epsilon_A} k^{-1/3}$ (new)
- Region III: localization in bulk for $\|A\|_{HS} \leq N^{-\epsilon_A}$ [S. Yang Yin '24]
- Region IV: localization at edge for $\|A\|_{HS} \leq N^{1/6-\epsilon_A} k^{-1/6}$ (new)

Additional observations

- We expect that the localization argument extends (with slight changes) into the gray region
- There is a gap of $N^{1/6}$ at the edge: more details later

In the bulk, we showed in [S. Yang Yin '24] that

- The bulk eigenvalue statistics for $\|A\|_{HS} \gg 1$ match those of a big ($DN \times DN$) GUE (consistent with prediction of BGS conjecture)
- The bulk eigenvalue statistics for $\|A\|_{HS} \ll 1$ match an ensemble with D iid copies of a GUE (consistent with prediction of Berry-Tabor conjecture)

Spectral decomposition I

Define the *Green's function* of H_Λ by

$$G(z) = (H_\Lambda - z)^{-1}, \quad (\operatorname{Im} z > 0).$$

The complex number $z =: E + i\eta$, is called the *spectral parameter*.
By spectral decomposition,

$$\operatorname{Tr}[\operatorname{Im} G(z) B \operatorname{Im} G(z) B^*] = \eta^2 \sum_{i,j} \frac{|\mathbf{v}_i^* B \mathbf{v}_j|^2}{|\lambda_i - z|^2 |\lambda_j - z|^2}, \quad (7)$$

where $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_{DN}$ are the eigenvalues of H_Λ.

Spectral decomposition II

Choose $B = E_a - D^{-1} \cdot I$ and $z = \gamma_k + i\eta_k$, where γ_k is the classical location of the eigenvalue λ_k and $\eta_k := N^{-2/3+\epsilon_L} k^{-1/3}$. By rigidity of eigenvalues,

$$
\mathbb{E}|\mathbf{v}_k^* E_a \mathbf{v}_k - D^{-1}|^2 \leq \eta_k^2 \, \mathbb{E} \, \mathrm{Tr}[\mathrm{Im}\, G(z)(E_a - D^{-1}I)\,\mathrm{Im}\, G(z)(E_a - D^{-1}I)]
$$

$$
= N\eta_k^2 \left(\mathbb{E}\mathcal{L}_{aa} - \frac{2}{D}\sum_b \mathbb{E}\mathcal{L}_{ab} + \frac{1}{D^2}\sum_{b,b'} \mathbb{E}\mathcal{L}_{bb'} \right)
$$

where $z_1 = z, z_2 = \overline{z}$ and we have defined

$$
\mathcal{L} := (L_{(1,2)} + L_{(2,1)} - L_{(1,1)} - L_{(2,2)})/4.
$$

$$
(L_{(1,2)})_{ab} := N^{-1}\,\mathrm{Tr}(G(z_1)E_a G(z_2)E_b),
$$

Key estimates for delocalization

Define the deterministic part $K_{(1,2)}$ of $L_{(1,2)}$ in terms of the deterministic part M of G by

$$K_{(1,2)} := (I - \widehat{M}_{(1,2)})^{-1} \widehat{M}_{(1,2)}, \quad (\widehat{M}_{(1,2)})_{ab} := D\langle M(z_1) E_a M(z_2) E_b \rangle.$$

To prove delocalization it is sufficient to show:

1. $\mathbb{E}L$ is close to K:

$$\| \mathbb{E}L_{(1,2)} - K_{(1,2)} \|_{HS} \leq N^{-1-c_L} \eta_k^{-2}, \quad z_1, z_2 \in \{z, \overline{z}\}$$
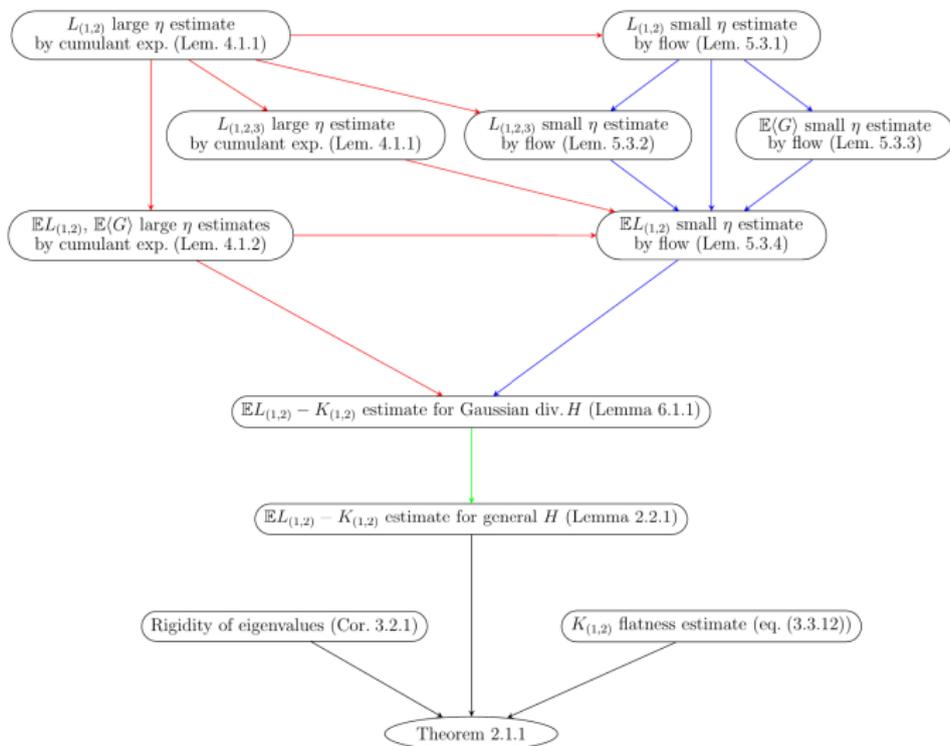
   up to N^ϵ, and

2. K is flat:

$$N\eta_k^2 \max_{a,b,a',b'} \left| (K_{(1,2)})_{ab} - (K_{(1,2)})_{a'b'} \right| = O\left(\frac{N^2 \eta_k^2}{\|A\|_{HS}^2} \right) = O(N^{-c_L}),$$

   for some constant $c_L > 0$.

For the second one, we need η_k to be small.

Outline of the proof

Step 1: Large η estimates

- We use the cumulant expansion method to compute the deterministic leading term of $\mathbb{E}L_{(1,2)}$ for large η

- The main goal is the estimate

  $$\| \mathbb{E}L_{(1,2)} - K_{(1,2)}\|_{HS} \leq N^{-1}\eta^{-2}\cdot(N^{-\epsilon_g}+N^{-\tau_e}), \quad \eta \gtrsim N^{-1/3+\tau_e},$$

  up to N^{ϵ}

- Requires a zero sum (self-energy renormalization) argument in the non-Gaussian case

- There are additional error terms that are too large for small η

Step 2: Small η estimates I

We use a flow method to extend the bound to small η. We allow H, z, Λ to depend on t.

1. The blocks H_a of H evolve according to the OU process

$$d(H_a)_t = -\frac{1}{2}(H_a)_t \, dt + \frac{1}{\sqrt{N}} d(B_a)_t,$$

where the B_a are independent Hermitian $N \times N$ Brownian motions
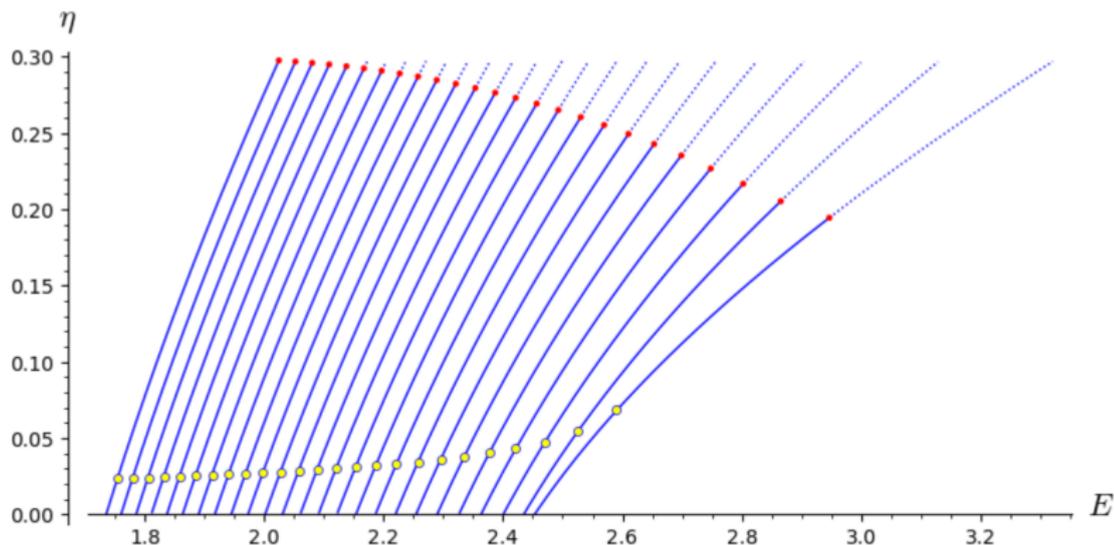
2. z_t and Λ_t evolve according to

$$\frac{d}{dt} z_t = -\frac{1}{2} z_t - \langle M_t \rangle, \quad \frac{d}{dt} \Lambda_t = -\frac{1}{2} \Lambda_t,$$

where M_t is the solution to the *Dyson matrix equation*
$M_t = (\Lambda_t - z_t - \langle M_t \rangle)^{-1}$.

Step 2: Small η estimates II

Here we plot some of the flows z_t for $A = N^{-0.1}I$, with $D = 4, N = 50$ (edge near $E = 2.7$). The flows run from time t_0 (red) until t_c.

Step 2: Small η estimates III

Using the rescaled quantities
$\widetilde{L}_{(1,2),t} = (t_c - t)L_{(1,2),t}, \widetilde{K}_{(1,2),t} = (t_c - t)K_{(1,2),t}$, we have

$$\frac{d}{dt} \mathbb{E}\widetilde{L}_t = \left(1 - \frac{1}{t_c - t}\right) \mathbb{E}\widetilde{L}_t + \frac{1}{t_c - t}(\mathbb{E}\widetilde{L}_t)^2 + \widetilde{\mathcal{E}}_t \qquad (8)$$

and

$$\frac{d}{dt} \widetilde{K}_t = \left(1 - \frac{1}{t_c - t}\right) \widetilde{K}_t + \frac{1}{t_c - t}\widetilde{K}_t^2.$$

After some additional steps, the difference satisfies

$$\frac{d}{dt}(\mathbb{E}\widetilde{L}_t - \widetilde{K}_t) = \frac{1}{t_c - t} \mathcal{T}_t(\mathbb{E}\widetilde{L}_t - \widetilde{K}_t) + \widehat{\mathcal{E}}_t$$

for a linear operator $\mathcal{T}_t$ such that $\|\mathcal{T}_t\|_{op} \leq 1 + o(1)$.

Step 2: Small η estimates IV

- Using the flow argument, we can prove the main estimate

$$\| \mathbb{E}L_{(1,2)} - K_{(1,2)}\|_{HS} \le N^{-1-c_L}\eta_k^{-2}, \quad z_1, z_2 \in \{z, \overline{z}\}$$
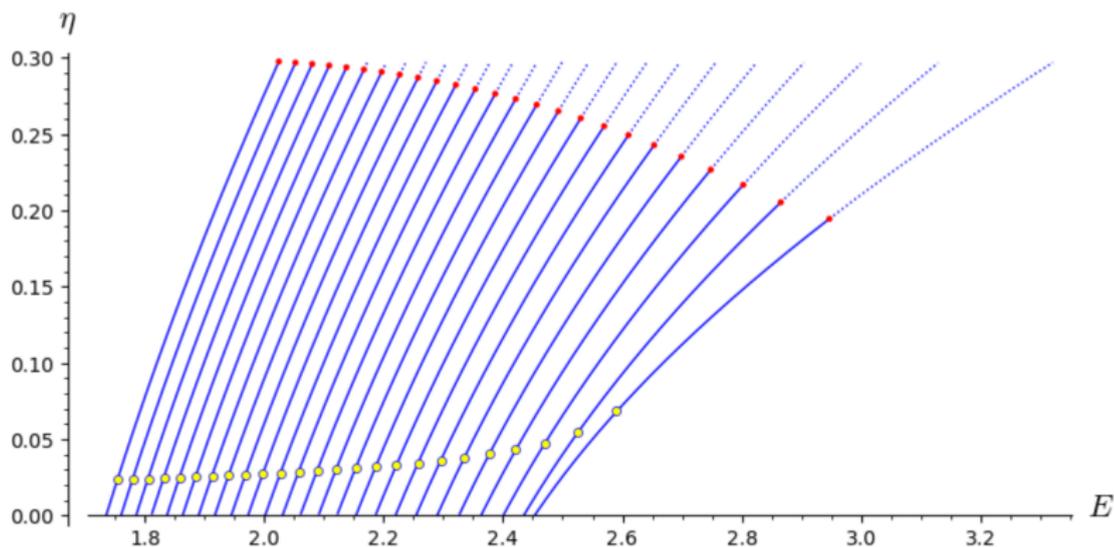
  (up to N^ϵ) at the correct scale, BUT

- This comes at the cost of replacing the original H with a Gaussian divisible $\widetilde{H}$ s.t.

$$\widetilde{H}_a \stackrel{d}{=} \sqrt{1 - N^{-\epsilon_g}}\widetilde{H}_a^{(0)} + N^{-\epsilon_g/2}\widetilde{H}_a^{(g)}, \qquad (9)$$

  where $\epsilon_g > 0$, $\widetilde{H}_a^{(g)}$ are i.i.d. GUE matrices, and $\widetilde{H}_a^{(0)}$ are independent Wigner matrices satisfying the moment conditions in Assumption ★

Step 2: Small η estimates V

Step 3: Green's function comparison

We need to show that the result holds for the original H.

- Under the assumptions on H, there exists a Gaussian divisible $\widetilde{H}$ s.t. the first three moments of H and $\widetilde{H}$ match exactly and the fourth moments $k + l = 4$ satisfy

$$\left| \mathbb{E}(H_{ij})^k (H_{ij}^*)^l - \mathbb{E}(\widetilde{H}_{ij})^k (\widetilde{H}_{ij}^*)^l \right| \lesssim N^{-2-\delta}.$$

- Using a standard Green's function comparison argument, we can show that the difference is small enough:

$$\mathbb{E}\langle G_1 E_a G_2 E_b \rangle - \mathbb{E}\langle \widetilde{G}_1 E_a \widetilde{G}_2 E_b \rangle \leq N^{-1-\delta} \eta^{-2},$$

up to N^ϵ.

Localization

Suppose H is a GUE, $k = O(1)$, and A satisfies Assumption $\bigstar$. We want to prove the following: if

$$\|A\|_{HS} \leq N^{1/6 - \epsilon_A} k^{-1/6}$$

for some $\epsilon_A > 0$, then there exists $c > 0$ such that

$$\max_{a \in [\![D]\!]} |\mathbf{v}_k^* E_a \mathbf{v}_k| \geq 1 - N^{-c}$$

with high probability.

Main estimate

Using another rigidity argument, the proof of localization reduces
to a bound of the form

$$\mathbb{E}\langle(\text{Im } G_0)\Lambda(\text{Im } G)\Lambda\rangle \leq (\text{Im } m)N^{-1+\epsilon}\|A\|_{HS}^2, \quad z = \gamma_k + \mathrm{i}N^{-2/3+c}, \tag{10}$$

where $G_0(z) = (H - z)^{-1}$.

We will use an iterated cumulant expansion argument. Terms that
appear in this argument are finite products of the following:

- A deterministic coefficient
- *Light weights* of the form $\langle(G - M)E_a\rangle$ for an index a
- Other loops equal to the normalized trace of products of
  finitely many factors of G, M, Λ, or E_j for an index j.

Furthermore, each term has two Λ entries.

Expansion rule

We apply the following expansion procedure to a term $\mathcal{T}$:

1. If $\mathcal{T}$ a loop $\langle \cdot \rangle$ containing both a G and a Λ, we choose the first G to the left of the first Λ, and split this factor using the identity $G = M - G(m + H)M$

2. We apply Gaussian integration by parts to the term with $-G(m + H)M$. We obtain $\mathbb{E}\,\mathcal{T} = \sum_j \mathbb{E}\,\mathcal{T}_j$ where each $\mathcal{T}_j$ is either smaller than $\mathcal{T}$ by a small factor, or the number of G factors in the expanded loop decreases.

3. If $\mathcal{T}_j$ is smaller than the error, or has no more loops containing both Λ and G, we stop the expansion. Otherwise, we send $\mathcal{T}_j$ to Step 1.

Expansion result

- Using this argument, we can compute:

$$\mathbb{E}\langle (\operatorname{Im} G_0)\Lambda(\operatorname{Im} G)\Lambda \rangle = -\frac{1}{4}(\langle m_{sc}\Lambda M\Lambda \rangle - \langle m_{sc}^*\Lambda M\Lambda \rangle$$
$$- \langle m_{sc}\Lambda M^*\Lambda \rangle + \langle m_{sc}^*\Lambda M^*\Lambda \rangle) + error$$
$$= \langle (\operatorname{Im} m_{sc})\Lambda(\operatorname{Im} M)\Lambda \rangle + error$$
$$= O\big((\operatorname{Im} m)N^{-1+\epsilon}\|A\|_{HS}^2\big),$$

for any $\epsilon > 0$.

- Note that the leading terms are too large individually, but there is some cancellation
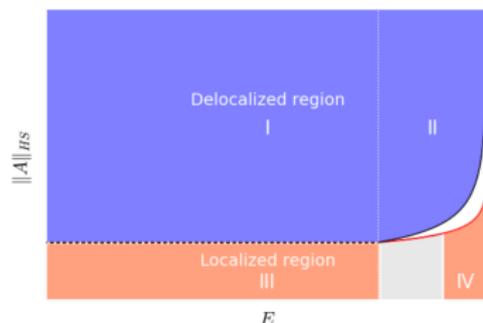
The gap

We suspect that the delocalization condition $\|A\|_{HS} \geq N^{1/3+\epsilon_A} k^{-1/3}$ is sharp:

- Numerical simulations
- The flatness condition on K

On the other hand, the current argument for localization does not extend into the gap:

- Stronger bound on $\mathbb{E}\langle (\mathrm{Im}\, G_0) \Lambda (\mathrm{Im}\, G) \Lambda \rangle$ is false
- A different argument using $\mathbb{E}\langle (\mathrm{Im}\, G_0)(\Lambda - a)(\mathrm{Im}\, G)(\Lambda - a) \rangle$ where $a = N^{-1}\|A\|_{HS}^2$

Extensions

The following are immediate areas of interest:

- Obtaining a sharp bound for localization at the edge
- Extending the localization result to $1 \ll k \ll N$ and general H

We can also consider general models where the M is more complicated:

- $H_1, \ldots, H_D$ with different variances
- Different matrices A_j representing the interactions between H_j, H_{j+1}

We can also consider a complementary model

$$\Delta + H := \begin{bmatrix} \Delta_1 & A & 0 & \cdots & A^* \\ A^* & \Delta_2 & A & 0 & 0 \\ 0 & A^* & \Delta_3 & A & \vdots \\ \vdots & 0 & A^* & \ddots & A \\ A & 0 & \cdots & A^* & \Delta_D \end{bmatrix} \tag{11}$$

Thank you!